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A Parametric Analysis of the Dynamic Lot-sizing Problem [1])

By *Knut Richter* and *Jozsef Vörös*

Abstract: This paper gives the analysis of the parametrized dynamic lot-size model. Three cases of parametrization are studied here: (i) setup and holding costs, (ii) demand vector, (iii) costs and demand. For all these cases the stability region of the parameters is found, i.e. it is shown for which parameters a solution generated by *Wagner-Whitin*'s algorithm remains valid. In the parametric analysis of the cases (i) and (ii) the parameter intervals are found by studying a simple system of inequalities. In the more complicated case (iii) the stability region of the two parameters is drawn by a computer program.

1. Introduction

In the dynamic lot-sizing problem (DLSP) the demand for the finished product occurs periodically and is known for T time periods in advance. Our models rely upon the assumption of linear inventory holding costs rather than the more general concave holding costs and holding costs remain constant in all time periods.

The DLSP is one of the best known standard model in OR/MS and the basic model of DLSP has been developed into many directions. Although the dynamic programming algorithm given by *Wagner* and *Whitin* (1958) for solving the uncapacitated DLSP can be considered as an effective one, heuristics are also studied frequently ([9], [6], [3]).

The theory of the multilevel lot-sizing problems is a useful generalization of the single-level DLSP, too ([5], [11], *Love* (1972), [4], [2], [1]) and relatively less effort has been devoted to the parametrical problems ([7], [8]).

The purpose of this paper is the investigation of the stability of an optimal schedule. Section 2 gives the parametric analysis of the objective function, while Section 3 gives that of the demand vector. Section 4 describes the two-parametric case.

2. Parametric analysis of the objective function

The DLSP is studied as

$$\sum_{t=1}^{T} (s \cdot \mathrm{sign}(x_t) + h \cdot y_t) \to \min; \tag{1}$$

$$y_{t-1} + x_t = y_t + d_t\,, \qquad t = 1, 2, \dots, T\,, \tag{2}$$

$$x_t \geqq 0; \quad y_t \geqq 0; \qquad t = 1, 2, \dots, T\,, \tag{3}$$

$$y_0 = y_T = 0\,. \tag{4}$$

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The inputs of DLSP are as usual:

T time horizon,

s = setup cost,

h = holding cost,

y_t = the inventory level at the end of period t,

x_t = the production level at period t,

$d_t > 0$ demand at period t.

In problem (1)—(4) total setup and holding cost is minimized subject to the satisfaction of the demand at each period.

The Wagner-Whitin algorithm is used to solve DLSP. Introducing

$$c(k, t) = s + h \cdot \sum_{r=k+1}^{t} (r - k - 1) \cdot d_t; \quad t = 1, 2, \dots, T; \quad k = 0, 1, \dots, \quad t - 1,$$

the following recursion is performed for all t:

$$f(0) = 0, \quad f(t) = \min\{c(k, t) + f(k): k < t\}, \quad t = 1, 2, \dots, T. \tag{5}$$

Let $f(k, t) = c(k, t) + f(k)$. The parameters $\{k(t)\}$, $t = 1, 2, \dots, T$ satisfying

$$f(t) = f(k(t), t),$$

called regeneration points, can be used to find an optimal solution for DLSP.

In [7] it is pointed out that the stability region of such $\{k(t)\}$, i.e. the set of all cost inputs (s, h) having a $\{k(t)\}$-solution, constitutes a convex cone with a given form. Now, however, the following cost structure is considered:

$$(\tilde{s}, \tilde{h}) = ((s, h) + a(\hat{s}, \hat{h})).$$

Let DLSP be solved for (s, h) with a given demand series and let $\{k(t)\}$ be a corresponding regeneration set.

Let $s(0)=h(0) = 0$,

$$\left.\begin{aligned} h(k, t) &= h(k) + \sum_{r=k+1}^{t} (r - k - 1) \cdot d, \\ s(k, t) &= s(k) + 1, \quad \text{and} \\ h(t) &= h(k(t), t), \\ s(t) &= s(k(t), t), \end{aligned}\right\} \tag{6}$$

furthermore $ds(k, t) = s(k, t) - s(t)$ and $dh(k, t) = h(k, t) - h(t)$ for $t = 1, 2, \dots, T$; $k = 0, 1, \dots, t - 1$.

It is the easy to find the set A of a's for which $\{k(t)\}$ remains valid.

Theorem 1. *The set A is given by the inequalities*

$$u(k, t) \leqq a \leqq v(k, t); \quad t = 1, 2, \dots, T, \; k = 0, 1, \dots, t - 1$$

$$s + as > 0; \quad h + ah > 0 \tag{7}$$

where $u(k, t)$ *equals to*

$$-(s \cdot ds(k, t) + h \cdot dh(k, t))/(\hat{s} \cdot ds(k, t) + \hat{h} \cdot dh(k, t)) \tag{8}$$

if the denominator of (8) *is positive, otherwise it is negative infinite.* $v(k, t)$ *also equals to* (8), *if the denominator is negative, otherwise it is positive infinite.*

Proof. It is enough to show the necessity of the conditions.
For costs $(\hat{s}, h)$ must be true that

$$\tilde{f}(k, t) \geqq \tilde{f}(t); \quad t = 1, 2, \dots, T; \qquad k = 0, 1, \dots, t-1 \tag{9}$$

where for cost inputs $(\tilde{s}, \tilde{h})$ $\tilde{f}(t)$ means the minimum cost of covering demands in the t-period problem while $\tilde{f}(k, t)$ means the same with the proviso that the last but one regeneration points is k. Using definitions in (6), if (s, h) has the same regeneration set $\{k(t)\}$, then (9) can be written as

$$\begin{aligned} \tilde{f}(k, t) &= (s + a\hat{s}) \cdot s(k, t) + (h + a\hat{h}) \cdot h(k, t) \\ &\geqq (s + a\hat{s}) \cdot s(t) + (h + a\hat{h}) \cdot h(t) \\ &= \tilde{f}(t), \end{aligned}$$

and thus the conditions

$$s * ds(k, t) + h * dh(k, t) \geqq -a(\hat{s} * ds(k, t) + \hat{h} * dh(k, t))$$

follow and conditions (7) have the same meaning. □

3. Parametric analysis of the demand vector

Let the demand vector $\tilde{\boldsymbol{d}}' = (\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_T)$ be expressed as

$$\tilde{d}_t = d_t + b \cdot \hat{d}_t; \qquad t = 1, 2, \dots, T;$$

and let the DLSP be solved for (s, h) and d_t, $t = 1, 2, \dots, T$.

To answer the question that for which b is $\{k(t)\}$ valid we introduce the following expressions:

$$\left.\begin{aligned} &\hat{h}(0) = 0; \\ &\hat{h}(k, t) = \hat{h}(k) + \sum_{r=k+1}^{t} (r - k - 1) \cdot \hat{d}_r; \\ &\hat{h}(t) = \hat{h}(k(t), t); \\ &d\hat{h}(k, t) = \hat{h}(k, t) - \hat{h}(t); \\ &t = 1, 2, \dots, T; \qquad k = 0, 1, \dots, t-1. \end{aligned}\right\} \tag{10}$$

Let $\tilde{f}(t)$ be the minimum cost of covering demands $\tilde{\boldsymbol{d}}$ with costs (s, h) in the t-period problem while $\tilde{f}(k, t)$ be the same with the proviso that k is a last but one regeneration point.

Theorem 2.
(i) *The set* B *of parameters* b *for which DLSP is solved by* $\{k(t)\}$ *is given by the following inequalities*:

$$\begin{aligned} &w(k, t) \leqq b \cdot h \leqq z(k, t); \quad t = 1, 2, \dots, T; \quad k = 0, 1, \dots, t-1; \\ &d_t + bd_t \geqq 0; \qquad t = 1, 2, \dots, T. \end{aligned} \tag{11}$$

(ii) *The equations*

$$\tilde{f}(t) = f(t) + b \cdot h \cdot \hat{h}(t); \qquad t = 1, 2, \dots, T \tag{12}$$

hold for all feasible b, *where* $w(k, t)$ *equals to*

$$-\big(f(k, t) - f(t)\big)/d\hat{h}(k, t) \tag{13}$$

if the denominator is positive, otherwise it is negative infinite. $z(k, t)$ *also equal to* (13) *if* $d\hat{h}(k, t)$ *is negative otherwise it is positive infinite.*

Proof. The statements are obviously true for a one-period DLSP. Since in the Wagner-Whitin algorithm the solution for a t-period problem is found with the help of the k-period problems, $k < t$, it is necessary to show how to derive the statement for the t-period case if the k-period case, $k < t$, is assumed to have been already proved. Let the t-period case be proved indirectly.

We assume that for feasible b there is some k such that

$$\tilde{f}(t) = \tilde{f}(k, t) < \tilde{f}(k(t), t) . \tag{14}$$

We remind that $\{k(t)\}$ is determined for the case d_t, $t = 1, 2, \dots, T$.

It follows from (14) that

$$\tilde{f}(k) + s + h \cdot \sum_{r=k+1}^{t} (r - k - 1) \cdot \hat{d}_r$$

$$< \tilde{f}(k(t)) + s + h \cdot \sum_{r=k+1}^{t} (r - k - 1) \cdot \hat{d}_r .$$

Assuming that (12) is true for $k < t$ and applying the definitions in (10) we can write:

$$f(k, t) + h \cdot b\hat{h}(k, t) < f(t) + h \cdot b \cdot \hat{h}(t) ,$$

or

$$f(k, t) - f(t) < h \cdot b \cdot h(k, t) .$$

In any case there is a contradiction either to the optimality of $\{k(t)\}$ for d_t, $t = 1, 2, \dots, T$, or to the definition of set B given in the formulation of the theorem.

The statement (ii) is an obvious consequence of statement (i);

$$\tilde{f}(t) = \tilde{f}(k(t)) + s + h \cdot \sum_{r=k(t)+1}^{t} (r - k - 1) \cdot \hat{d}_r$$

$$= f(t) + h \cdot b \cdot \hat{h}(t) . \; \square$$

4. Two-parametric analysis

Our task in this section is to determine the set AB of (a, b)'s for which the solution $\{k(t)\}$ of DLSP with inputs $(s, h, \boldsymbol{d})$ is valid. The inputs of DLSP are: $\tilde{s}, \tilde{h}, \tilde{\boldsymbol{d}}$.

Theorem 3. *The set* AB *is given by the following system of (partly nonlinear) inequalities:*

$$p(k, t, a) \leqq b \leqq q(k, t, a); \tag{15}$$

$$t = 1, 2, \dots, T; \qquad k = 0, 1, \dots, t - 1;$$

$$u(k, t) \leqq a \leqq v(k, t); \tag{16}$$

$$t = 1, 2, \dots, T; \qquad k = 0, 1, \dots t - 1;$$

$$s + a\hat{s} > 0; \qquad h + a\hat{h} > 0; \qquad d_t + b\hat{d}_t > 0 ; \qquad t = 1, 2, \dots, T; \tag{17}$$

where $p(k, t, a)$ *equals to*

$$-\left((s + a\cdot\hat{s})\cdot ds(k, t)/(h + b\cdot\hat{h}) + dh(k, t)\right)/d\hat{h}(k, t) \tag{18}$$

if $d\hat{h}(k, t)$ *is positive, otherwise it is negative infinite and* $q(k, t, a)$ *also equals to* (18) *if* $d\hat{h}(k, t)$ *is negative, otherwise it is positive infinite; the values* $u(k, t)$ *and* $v(k, t)$ *are determined as in Theorem* 1 *provided that* $d\hat{h}(k, t) \neq 0$, *otherwise they are negative or positive infinite respectively.*

Proof. For inputs $(\tilde{s}, \tilde{h}, \tilde{\mathbf{d}}')$ it must be true that

$$\tilde{f}(k, t) \geqq \tilde{f}(t); \qquad t = 1, 2, \ldots, T; \qquad k = 0, 1, \ldots t-1\,. \tag{19}$$

If $(s, h, \mathbf{d}')$ and $(\tilde{s}, \tilde{h}, \tilde{\mathbf{d}}')$ have the same regeneration set $\{k(t)\}$ then we can write (19) as

$$(s + a\cdot\hat{s})\cdot s(k, t) + (h + a\cdot\hat{h})\cdot\left(h(k, t) + b\cdot\hat{h}(k, t)\right)$$
$$\geqq (s + a\cdot\hat{s})\cdot s(t) + \left(h(t) + b\cdot\hat{h}(t)\right)(h + b\cdot\hat{h})$$

or otherwise

$$(s + a\cdot\hat{s})\cdot ds(k, t) + (h + a\cdot\hat{h})\cdot dh(k, t)$$
$$\geqq -(h + a\cdot\hat{h})\cdot b\cdot d\hat{h}(k, t)\,.$$

Then, if $d\hat{h}(k, t) \neq 0$, conditions (15) directly follow, while if $dh(k, t) = 0$ then it must be valid that

$$s\cdot ds(k, t) + h\cdot dh(k, t)$$
$$\geqq -a\cdot\left(\hat{s}\cdot ds(k, t) + \hat{h}\cdot dh(k, t)\right)$$

and from this assumptions (16) are necessary.
The sufficiency of the conditions can be seen easily. □

Example. Let $T = 3$; $s = 5$, $\hat{s} = 1$; $h = 2$, $\hat{h} = -1$; $\mathbf{d} = (3, 2, 1)$ and $\hat{\mathbf{d}} = (-1, 0, 1)$. Then the following parameters will be determined (compare Table 1).
Then $dh(k, t) \neq 0$ for index pairs $(k, t) = (0, 3)$ and $(2, 3)$. For these assumptions the inequalities

$$-7/(a-2) - 4 \leqq b\,,$$
$$1 \geqq b$$

hold, $d\hat{h}(k, t) = 0$ and the denominator of (8) is not equal to zero for the index pair $(k, t) = (1, 2)$ and the constraint from this is

$$-1/3 \leqq a\,.$$

Table 1

$t =$	1	2	3
$k = 0$	0	0	2
1	—	0	1
2	—	—	0
$h(t)$	0	0	1

The values of $\hat{h}(k, t)$

$t =$	1	2	3
$k = 0$	0	2	4
1	—	0	1
2	—	—	2
$h(t)$	0	2	1

The values of $h(k, t)$

$t =$	1	2	3
$k = 0$	1	1	1
1	—	2	2
2	—	—	2
$s(t)$	1	1	2

The values of $s(k, t)$

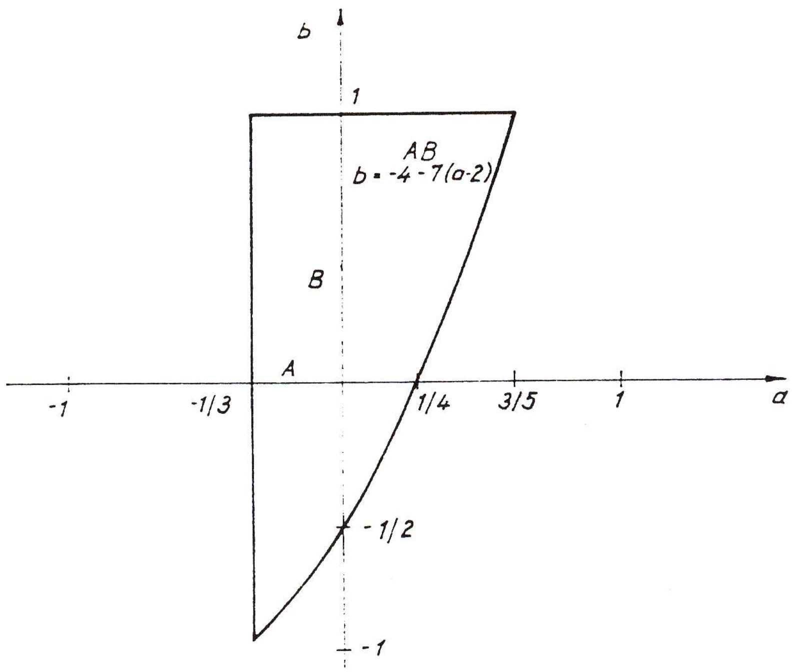

Fig. 1. Set AB.

Table 2

$T =$	3	5	7	10	15
computation time on IBM-XT (min:sec)	0:7	0:19	0:33	1:09	2:41

The other constraints are redundant.

The solution of the problem is represented by Fig. 1. Table 2 contains the computation time of some examples whose cost inputs are as above while $\boldsymbol{d}$ and $\hat{\boldsymbol{d}}$ are generated randomly. The computation time includes the printing at the monitor out of all constraints describing the set AB.

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Kurzfassung

Die Arbeit liefert eine Analyse des parametrisierten dynamischen Losgrößenmodells. Drei Fälle der Parametrisierung werden untersucht: (1) Fix- und Lagerkosten, (2) Bedarfsvektor, (3) Kosten und Bedarf. Für alle drei Fälle werden Stabilitätsbereiche der Parameter ermittelt, d. h., es wird gezeigt, für welche Parameter eine durch den Wagner-Whitin-Algorithmus berechnete Lösung gültig bleibt.

In den Fällen (1) und (2) werden die Parameterintervalle aus einfachen Systemen von Ungleichungen abgeleitet. Im komplizierten Fall (3) wird der Stabilitätsbereich von zwei Parametern mit Hilfe eines Rechnerprogramms dargestellt.

Резюме

Исследована параметрическая динамическая задача управления запасами. Рассмотрены три частных случая: (1) для постоянных издержек и издержек хранения, (2) для вектора спроса, (3) для издержек и спроса. Для этих случаев найдена область устойчивости параметров, т.е. показано для каких значений параметров остаются верными решения, найденные алгоритмом *Вагнера* и *Вайтина*. Анализ параметров для случаев (1) и (2) произведен исследованием простых систем неравенств. Область устойчивости для более сложного случая двух параметров (3) определена при помощи программы на персональном компьютере.

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Authors' addresses:

Prof. Dr. K. Richter
Technical University of Karl-Marx-Stadt
Dept. of Economic Sciences
P.O. Box 964
9010 Karl-Marx-Stadt
German Democratic Republic

Prof. Dr. J. Vörös
Janus Pannonius University of Science
Faculty of Economics
Pécs
Hungary