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STABILITY OF A TWO-STAGE PRODUCTION AND INVENTORY MODEL

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ABSTRACT

The serial assembly model is considered. The problem is formulated to find sets of cost inputs for which solutions found by a recursion procedure remain valid. For simplicity a solution of this problem is provided for the two-stage problem. The paper shows that the stability region of cost inputs forms a convex cone in $\mathbb{R}^4$ and consists of a system of linear inequalities. An algorithm is provided to compute the parameters of this cone and several cases of changing only two parameters are displayed graphically.

1. THE MULTI-STAGE SERIAL ASSEMBLY PROBLEM

This well-known model has been studied by many authors (Zangwill [1], Love [2], Lambrecht et al. [3], Graves [4], Blackburn and Millen [5], Chand [6], Afentakis et al. [7]). A comprehensive formulation of the problem is given below:

$$\min \sum_{s=1}^{S} \sum_{t=1}^{T} (c_s \text{sign} x_{st} + h_s I_{st})$$

Subject to:

$$\begin{array}{lll} I_{st} = I_{s,t-1} + x_{st} - x_{s+1,t} & s=1,...,S,\ t=1,...,T & \\ x_{st} \geqslant 0, \quad I_{st} \geqslant 0 & s=1,...,S,\ t=1,...,T & (P) \\ I_{s0} = I_{sT} = 0 & s=1,...,S & \\ x_{s+1,t} = d_t & t=1,...,T & \end{array}$$

This model has the following economic interpretation: A production–inventory system consists of S facilities in series. The input to facility $s+1$ comes from facility s. Facility S produces assemblies which are used to supply the customer demand d_t for periods $t=1,...,T$. All facilities may carry inventories. It is assumed that production and shipment are instantaneous and that one unit of production at facility $s+1$ requires one unit of input from facility s. Backlogging of demand is not allowed. Let x_{st} denote the production at facility s in period t, and the cost of production be provided by $c_s$sign x_{st}. The stock at the end of t-th period at facility s is denoted by I_{st}. The holding cost is provide by $h_s I_{st}$. It is easy to see that the model has been designed to determine a minimum cost production and inventory strategy for S facilities and T periods which meets the demand of all periods.

It is also well known that there are optimal solutions for this type of model which fulfil

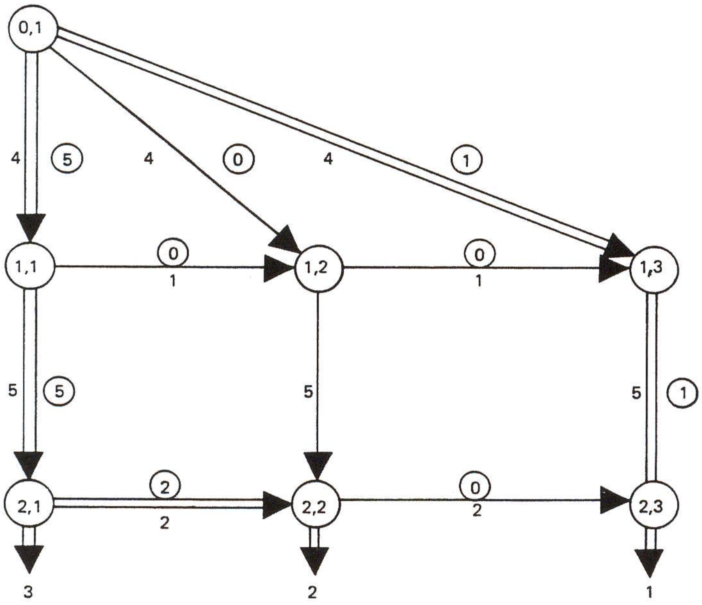

Fig. 1. Network interpretation of the two-stage problem.

$I_{s,t-1}x_{st}=0$

and

$x_{s+1,t}>0 \quad \text{if } x_{st}>0.$

Based on these properties algorithms have been designed which employ the idea of dynamic programming. The question is what will happen if the data, i.e., cost inputs, change. If the stability region of a given optimal solution can be determined and we are able to indicate all inputs, for which the solution is optimal, it is possible to answer such a question. In other words, a decision can be prepared whether input changes force the strategy of production and inventory control to be revised. The problem under study has been considered for the one-stage case (cf. [8,9,10,11]). In this sense the results below may be regarded as a generalization. A complete solution for the two-stage model will be provided in the next two sections. Let an example be regarded before studying the algorithm.

Example

Let $T=3$, $d_1=3$, $d_2=2$, $d_3=1$, $c_1=4$, $c_2=5$, $h_1=1$ and $h_2=2$. The problem can be viewed as a network problem of the following type (see Fig. 1). An optimal solution to the problem is provided by a flow from node (0,1) to the nodes (2,1)–(2,3), if the flow satisfies the demand and minimizes the cost along the arcs. The flow drawn in small circles in the figure shows a feasible solution with cost $4+4+5+5+2\times2=22$. It will be seen later that this is not the optimal solution.

2. SOLVING THE TWO-STAGE PROBLEM

Chands's algorithm will be studied for the case of two facilities. It employs the following idea:

Every solution divides the periods into such groups for which the production at the first stage is performed only in the first period of the group.

The solution provided in Fig. 1 divides the three periods into two groups, $\{1,2\}$ and $\{3\}$.

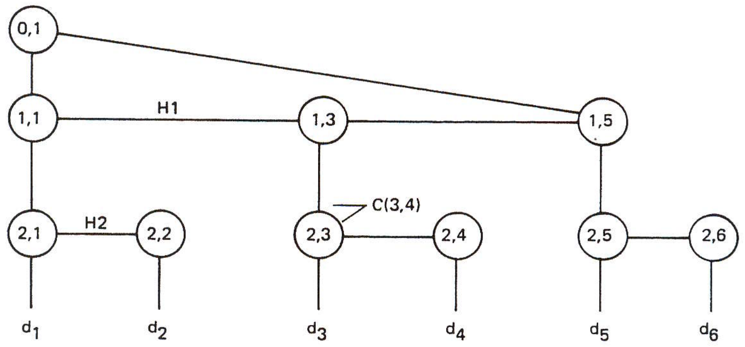

Fig. 2. Network interpretation of a complex solution.

In all these groups subgroups can be found for which the production at the second stage is performed only in the first period of the subgroup. In the example of Section 1 the groups and subgroups coincide. In more complex problems the situation may be similar to that in Fig. 2.

The algorithm will, on the one hand, find the optimal decomposition of groups of arbitrary length into subgroups, and on the other hand, on the basis of this information determine the optimal decomposition of the periods 1,..., T into groups. Then the solution of the problem will be obviously on hand.

Let arbitrary periods $r \leqslant t$ be regarded and let $C(r,T)$ denote the minimal sum of holding costs for both stages and production cost at the second stage for the periods $r,r+1,...,t$. $C(r,t)$ can be determined by the typical dynamic programming philosophy:

$$c(r,t) = \min_{r \leqslant n \leqslant t} \{c_2 + h_2 \sum_{j=r+1}^{n} d_j(j-r) + h_1(n+1-r) \sum_{j=n+1}^{t} d_j + C(n+1,t)\}. \quad (1)$$

The optimal period n is denoted by $n(r,t)$. In this recursion

$$H2(r,n) = \sum_{j=r+1}^{n} d_j(j-r)$$

provides the stock at the second stage associated with storing the items needed in $r+1,...,n$, and

$$H1(r,n,t) = (n+1-r) \sum_{j=n+1}^{t} d_j$$

covers the stock at the first stage associated with storing items needed at the second stage in periods $r,r+1,..., n$. In Fig. 2 these parameters show the values $H2(1,2) = d_2$ and $H1(1,2,4) = 2(d_3+d_4)$.

Now let $f(t)$ denote the minimal total cost for the problem with the periods $t,t+1,..., T$. Then, obviously,

$$f(r) = \min_{r \leqslant n \leqslant t} \{c_1 + C(r,n) + f(n+1)\} \quad (2)$$

holds according to the dynamic programming approach. The optimal parameter n will be denoted by $n(r)$.

The complete algorithm can be described in computer language, as shown in Fig. 3. The parameters $H1$ and $H2$ are included in the output since they will be used for the stability analysis in the next section.

The application of the algorithm AL to the example from Section1 provides the outputs

$$\{n(r)\} = (3,3,3)$$

$$\{n(r,t)\} = \begin{bmatrix} 1 & 2 & 3 \\ - & 2 & 3 \\ - & - & 3 \end{bmatrix}.$$

$$\{H1(1,n,t)\} = \begin{bmatrix} 0 & 2 & 3 \\ - & 0 & 2 \\ - & - & 0 \end{bmatrix} \qquad \{H1(2,n,t)\} = \begin{bmatrix} - & - & - \\ - & 0 & 1 \\ - & - & 0 \end{bmatrix}$$

$$\{H2(r,t)\} = \begin{bmatrix} 0 & 2 & 4 \\ - & 0 & 1 \\ - & - & 0 \end{bmatrix}$$

$$f(r) = (17,11,9).$$

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Algorithm AL

input: $c_1, c_2, h_1, h_2, d_1,\ldots, d_T$
$f(T+1):=0$
for $r=T$ downto 1 do
begin for $t=r$ to t do
   begin determine $H2(r,t)$
      for $n=r$ to $t$ do determine $H1(r,n,t)$
      determine $C(r,t)$ and $n(r,t)$
   end
   $f(r):=\min\{c_1+C(r,t)+f(t+1):r\leqslant t\leqslant T\}$
   determine $n(r)$
end
output: $n(r), n(r,t), H1(r,n,t), H2(r,t), f(r)$.
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Fig. 3. Algorithm AL.

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Algorithm AL1

input: $n(r), n(r,t), 1\leqslant r\leqslant t\leqslant T$
$r:=1$
while $r\leqslant T$ do
begin $t:=n(r)$
   $x_{1r}:=\sum_{j=r}^{t} d_j$
   for $j=r+1$ to $t$ do $x_{1j}:=0$
   $n:=r$
   while $n\leqslant t$ do
   begin
      $t1:=n(n,t)$
      $x_{2n}:=\sum_{j=n}^{t1} d_j$
      for $j=n+1$ to $t1$ do $x_{2j}:=0$
      $n:=t1+1$
   end
   $r:=t+1$
end
output: $x_{st}, s=1,2, t=1,\ldots, T$
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Fig. 4. Algorithm AL1.

Thus the minimal value of the example is given by $f(1)=17$. The solution of the problem can be found by the algorithm stated in Fig. 4.The application of algorithm AL1 generates the solution

$$x=\begin{pmatrix}6 & 0 & 0\\ 6 & 0 & 0\end{pmatrix},$$

which really is associated with the cost$=4+5+(2\times3)+(2\times1)=17$.

3. DETERMINING THE STABILITY REGION

The question now is how c_1, c_2, h_1, h_2 may change without affecting the values $n(r)$ and $n(r,t)$ and, therefore, leaving the solution generated by algorithm AL1 valid. Let us note that the set of c_1, c_2, h_1, h_2 answering this question may be only a subset of the real stability region. This can happen since the actual solution may also be derived from other $n(r)$ and $n(r,t)$. Finding the complete stability region is therefore a problem of studying all possible $n(r)$ and $n(r,t)$ in the case of nonunique parameters. This will be left for further research.

Let the expressions of the right hand side of (1) and (2) be denoted by $c(r,n,t)$ and $f(r,n)$, respectively. Then the following theorems can be formulated.

Theorem 1. The optimal parameters $n(r)$ and $n(r,t)$ remain valid for all c_1, c_2, h_1, h_2 such that

$$\begin{aligned} C(r,t)&=C(r,n(r,t),t)\leqslant C(r,n,t), && r\leqslant n\leqslant t,\\ f(r)&=f(r,n(r))\leqslant f(r,n), && r\leqslant n\leqslant T. \end{aligned} \tag{3}$$

Theorem 2. The set of all c_1, c_2, h_1, h_2 satisfying the inequalities (3) form a convex polyhedral cone in the $\mathbb{R}^4$.

Theorem 1 certainly needs no more arguments to be accepted. In order to discuss the second theorem let the following new parameters be introduced:

$K(s,r,t)=$ optimal number of setups at the s-th stage for the periods $r,\ldots, t$, and

$H(s,r,t)=$ optimal stock at the s-th stage for the periods $r,\ldots, t$.

These parameters can be derived from the outputs of algorithm AL. Then

$$\begin{aligned} C(r,n,t)=&\,h_1H1(r,n,t)+h_2H2(r,n)\\ &+c_2(1+K(2,n+1,t))\\ &+h_1H(1,n+1,t)+h_2H(2,n+1,t) \end{aligned} \tag{4}$$

and

$$\begin{aligned} f(r,n)=&\,c_1(1+K(1,n+1,T))\\ &+c_2(K(2,r,n)+K(2,n+1,T))\\ &+h_1(H(1,r,n)+H(1,n+1,T))\\ &+h_2(H(2,r,n)+H(2,n+1,T)) \end{aligned} \tag{5}$$

hold. It is clear that the application of the expressions (4) and (5) to the inequalities (3) provides a system of linear inequalities with the variables c_1, c_2, h_1, h_2. Since that system is obviously homogeneous the set must be a convex polyhedral cone.

The system (3) is not suitable for computation. Let, therefore, the parameters $K(s,r,t)$ and $H(s,r,t)$ be determined by the algorithm depicted in Fig. 5. This algorithm finds systematically all the values needed. Its application to the data of the output of the algorithm AL provides the following information:

$$\{K(1,r,T)\} = (1,1,1)$$

$$\{K(2,r,t)\} = \begin{pmatrix} 1 & 1 & 1 \\ - & 1 & 1 \\ - & - & 1 \end{pmatrix}$$

$$\{H(1,r,t)\} = \begin{pmatrix} 0 & 0 & 0 \\ - & 0 & 0 \\ - & - & 0 \end{pmatrix}$$

$$\{H(2,r,t)\} = \begin{pmatrix} 0 & 2 & 4 \\ - & 0 & 1 \\ - & - & 0 \end{pmatrix}.$$

Now let the inequalities (3) be transformed using the new symbols. Then

$$\begin{aligned} &h_1[H1(r,n,t)+H(1,n+1,t)-H(1,r,t)] \\ &\quad +h_2[H2(r,n)+H(2,n+1,t)-H(2,r,t)] \\ &\geqslant c_2[K(2,r,t)-1-K(2,n+1,t)] \end{aligned} \qquad (6)$$

and

$$\begin{aligned} &h_1[H(1,r,n)+H(1,n+1,T)-H(1,r,t)] \\ &\quad +h_2[H(2,r,n)+H(2,n+1,T)-H(2,r,t)] \\ &\geqslant c_1[K(1,r,T)-1-K(1,n+1,T)] \\ &\quad +c_2[K(2,r,T)-K(2,r,n)-K(2,n+1,T)] \end{aligned} \qquad (7)$$

will be obtained for all $1 \leqslant r \leqslant n \leqslant t \leqslant T$.

Hence, the following theorem will summarize the results.

Theorem 3. The stability region of the parameters $n(r)$ and $n(r,t)$ is correctly described by the inequalities (6) and (7).

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Algorithm KS

input: n(s), n(r,t), H1(r,n,t), H2(r,t), $1 \leqslant r \leqslant t \leqslant T$
for r=t downto 1 do
begin n:=n(r)
  K(1,r,T):=1+K(1,n+1,T)
  for t=r to T do
  begin t1:=r
    HA:=HB:=KA:=0
    while t1 $\leqslant$ t do
    begin n1:=n(t1,t)
      HA:=HA+H1(t1,n1,t)
      HB:=HB+H2(t1,n1)
      KA:=KA+1
      t1:=n1+1
    end
    H(1,r,t):=HA
    H(2,r,t):=HB
    K(2,r,t):=KA
  end
  K(2,r,t):=K(2,r,n)+K(2,n+1,T)
  H(1,r,t):=H(1,r,n)+H(1,n+1,T)
  H(2,r,T):=H(2,r,n)+H(1,n+1,T)
end
output: K(1,r,T), K(2,r,t), H(1,r,t), H(2,r,t), $1 \leqslant r \leqslant t \leqslant T$
```

Remark: All parameters are supposed to be set zero in the algorithm if second index *r* is greater than the third one *t*.

Fig. 5. Algorithm KS.

These inequalities are for the example of Section 1 of the following form:

$$\begin{aligned} 2h_1-2h_2 &\geqslant -c_2 \\ 3h_1-3h_2 &\geqslant -c_2 \\ h_1-\ h_2 &\geqslant -c_2 \\ h_2 &\geqslant -c_1-c_2 \\ -3h_2 &\geqslant -c_1-c_2 \\ -2h_2 &\geqslant -c_1-c_2 \\ -\ h_2 &\geqslant -c_1-c_2 \end{aligned} \qquad (8)$$

While it seems to be hard to interprete the full system (8), several special cases may be of interest (cf. Figs. 6–9).

(i) Production cost inputs remain constant, i.e., $c_1=4$ and $c_2=5$, then $h_2 \leqslant 3$, $2(h_2-h_1) \leqslant 5$, $3(h_2-h_1) \leqslant 5$.

(ii) Holding cost inputs remain constant, i.e., $h_1=1$ and $h_2=2$, then $c_1+c_2 \geqslant 6$, $c_2 \geqslant 3$.

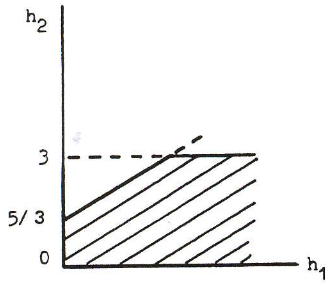

Fig. 6. c_1, c_2 fixed.

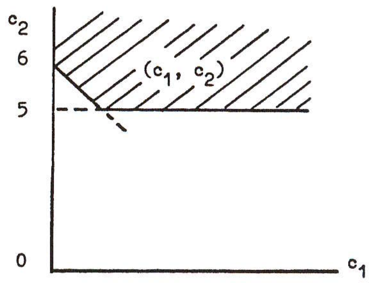

Fig. 7. h_1, h_2 fixed.

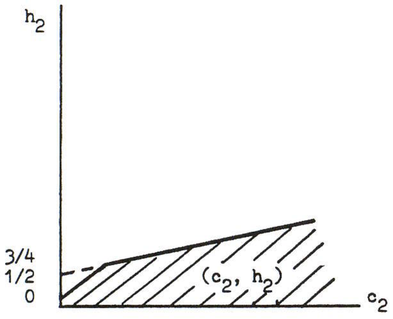

Fig. 8. c_1, c_2 fixed.

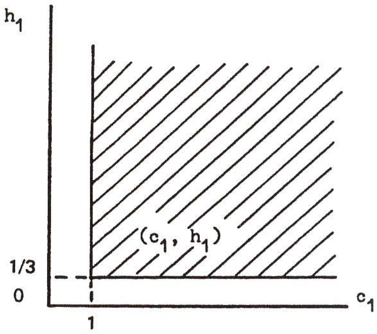

Fig. 9. c_2, h_2 fixed.

(iii) Cost inputs of the first stage remain constant, i.e., $c_1=4$ and $h_1=1$, then $3h_2-c_2\leqslant 3$, $2h_2-c_2\leqslant 2$, $h_2-c_2\leqslant 1$.

(iv) Cost inputs of the second stage remain constant, i.e., $c_2=5$ and $h_2=2$, then $c_1\geqslant 1$, $3h_1\geqslant 1$.

These four cases illustrate different cuts into the stability region by fixing the values of two cost inputs. Thus, subsets of the stability region occur, which can be graphically displayed. Such a partial look at system (8) seems to be the only way to use the "four-dimensional" information for economic analysis.

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